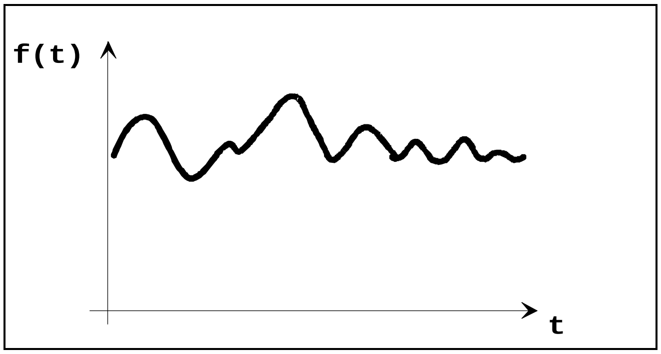
SECTION 1
INTRODUCTION

If we were to examine the amplitude versus time history of a process variable such as pressure, temperature, flow or force, we would in all likelihood observe variations in amplitude over time (Figure 1). If our objective is to measure this time-varying phenomenon with a digital sampled data acquisition system, then there is one central issue, which must be addressed. Since information is lost when a continuum is represented with a finite number of samples, the central issue relates to how many discrete samples must be acquired. Alternatively, what is the required sampling rate?



a. Time History of Process Variable

Our intent with this document is to answer the question of "How fast must I sample?" To answer this requires first that we quantify the band of frequencies that we are interested in (i.e., establish the desired bandwidth). Once bandwidth has been quantified, *sampling rate can be established based on the desired bandwidth and on the highest frequency beyond which there is no detectable energy*. In Section 2 we present bandwidth as the basis for establishing the sampling rate parameters. Section 3 discusses the relationship between sampling rate and anti-alias filter characteristics and presents a method to compute sampling rate based upon the specific application. Finally, Section 4 introduces processing sampled data using digital filtering.

SECTION 2
ESTABLISHING BANDWIDTH

We hypothesize that if we decompose the time varying input signal into its frequency contents (i.e., represent the signal with sinusoidals using Fourier Transforms), we would observe that the complex signal consists of one or more discrete frequencies. Some of these may be attributable to either the process dynamics or to noise. If they are noise, our objective will be to discriminate against them (i.e., filter or remove them from the frequencies of interest) such that they do not affect the individual samples. Generally speaking, noise is considered to be of higher frequency than the process dynamics thus enabling us to use a low-pass filter to pass the lower frequencies of interest and attenuate the higher noise frequencies.

The lower frequencies attributable to the process dynamics may or may not be of interest depending upon the measurement objectives. For example, if we are interested in the average behavior of a process, we need only take enough samples to average out the variations. However, since there may be multiple low frequencies, establishing the true average for any time interval is complicated. Alternatively, we can use a hardware implemented low-pass filter with a cutoff frequency near zero. This, however, will make the system overly sluggish and may compromise overall measurement accuracy as well as compromise out-of-limit detection schemes. For example, a 1 Hertz filter has a 1-second time constant. If a step input is applied the filter's output will respond exponentially and will require several time constants before reaching final value.

Oftentimes the low frequency variations are of interest and the objective is to preserve both the frequency and amplitude of these. Consider for example an engine-dynamometer test stand with observed low frequency variations in the torque measurement. Are the variations in torque caused by non-uniform fuel burning, non-uniform fuel delivery or some other facet of the engine's performance, or are these perhaps due to the torque measurement process itself? If we rely upon time averages, the variations may go unnoticed.

Establishing the band of frequencies of interest is the basis for establishing all sampling parameters including filter characteristics, sampling rates and post sampler operations such as digital filtering. Accordingly, bandwidth must be established based on measurement objectives, knowledge of the process dynamics and knowledge of the various measurement techniques and their response characteristics. If the measurement objectives include establishing correlation between different measured variables, performing closed-loop process control, analyzing the time amplitude variations or detecting out-of-limit conditions, then simple averaging should not be used. Instead, care must be taken not to distort the amplitude-frequency characteristics within the band of interest.

SECTION 3
ESTABLISHING SAMPLING RATE

Consider the input function (Figure 2a). As shown, the function varies with time. The equivalent frequency domain representation of this is shown in Figure 2b. Here it is shown that the function has energy at different frequencies extending up to f_x. Beyond f_x no energy is present. Based on an analysis of the process dynamics and in consideration of the measurement objectives, the frequency band of interest has been predetermined to lie from zero to f_c. Frequencies beyond f_c are considered unwanted. *The measurement objective is to represent the input function f (t) with a finite set of samples which are adequate to preserve both the frequency and amplitude information between zero and* f_c.

Ideally, we would like to process the input function through a hardware implemented low-pass filter which passed all frequencies between 0 and f_c without attenuation and had infinite attenuation for all frequencies greater than f_c (Figure 3a).

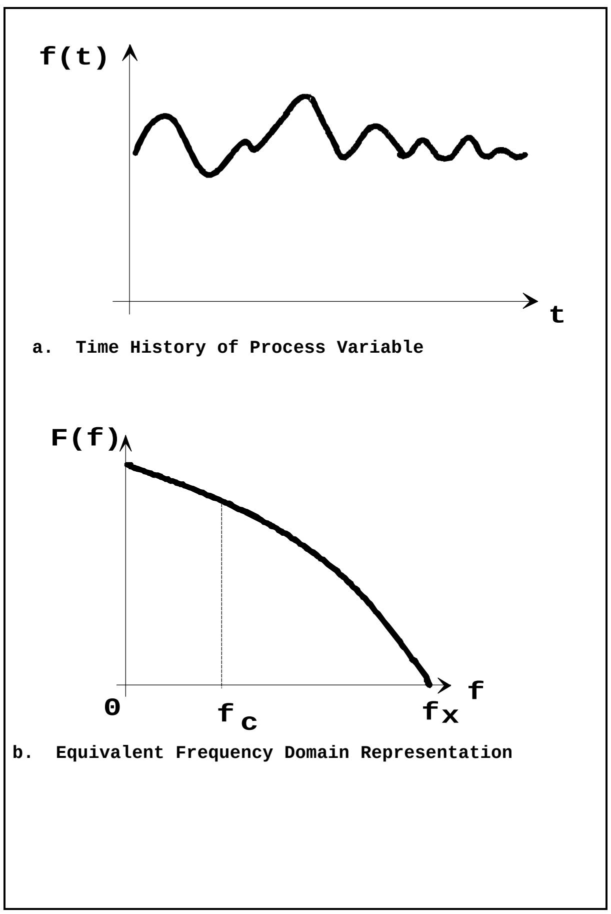


Figure 2. Time and Frequency Domain Representations of Input Function

Practically speaking, the low-pass filter will attenuate all frequencies past the filter's cutoff frequency (here the filter's cutoff is selected to be f_c) at a finite rate depending upon the filter's attenuation characteristics. Figure 3b illustrates the attenuation characteristics for several different filters.

Since attenuation is stated in terms of dB/Octave, we can calculate the frequency f_c^* corresponding to any desired attenuation level as follows:

No. Of Octaves, N = (Desired Attenuation, dB) / (Filter Rolloff, dB)

For example, if we are using a digital data acquisition system, which uses an m-bit analog-to-digital converter, then we cannot distinguish any energy, which is less than the converter's resolution. We can use this as a measure of desired attenuation level.

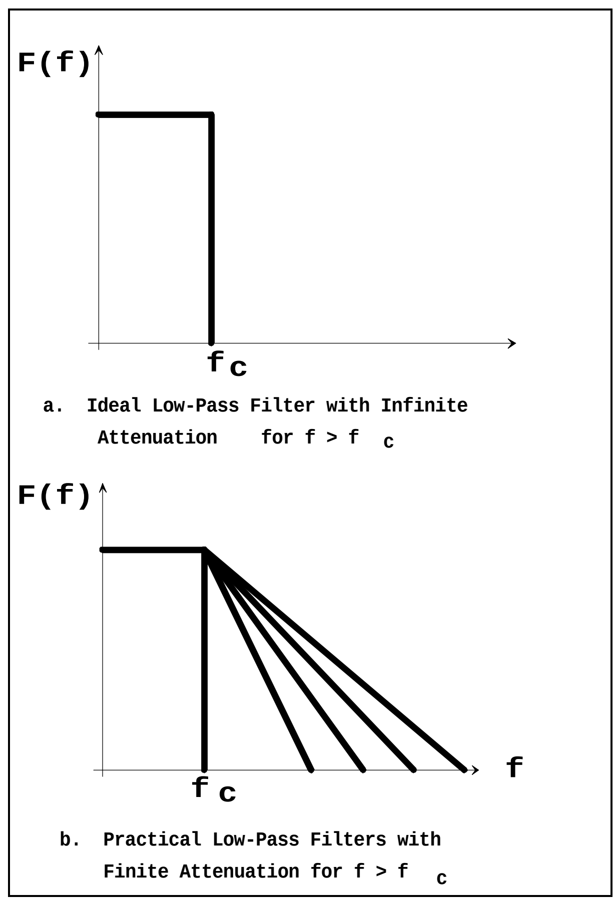


Figure 3. Attenuation Characteristics for Low-Pass Filters

That is,

Desired Attenuation Level, dB = $20\log(2^m)$

Using N computed above, the frequency f_c^* beyond which there is no detectable energy can be computed as:

$$f_c^* = 2^N f_c$$

From this it can be seen that the practical filter more closely approximates the ideal filter for values of N that are small. That is, the greater the attenuation rate, the closer f_c^* will be to f_c.

The principal criterion to use in selecting the sampling rate f_s is to ensure that the band of frequencies from zero to f_c is not distorted as a consequence of sampling. That is, no higher frequencies are aliased into this fundamental interval. However, since we are not concerned about frequencies between f_c and f_c^*, we can select an f_s such that distortion occurs in this interval.

To compute sampling rate f_s, we first calculate the folding frequency, f_n, which is the midpoint between fc and f_c*. Thus

$$f_n = ½(f_c + f_c^*)$$

To ensure the integrity of the fundamental interval, the sampling frequency, f_s, must be at least twice f_n. That is,

$$f_s \quad 2f_n$$

Figure 4 illustrates the relationship between f_c, f_n, f_c^* and f_s. As shown, the fundamental interval, zero to f_c, remains as it was prior to sampling and thus has not been distorted. However, distortion is present beyond f_c.

The infinite sidelobes that result from sampling can be viewed in terms of a Frequency Folding Diagram (Figure 5). Because of the non-uniqueness of equally spaced sampled data, any energy we observe in the fundamental interval of zero to f_c can be the result of energy at that frequency or can be the result of a higher frequency that is aliased down to the fundamental interval.

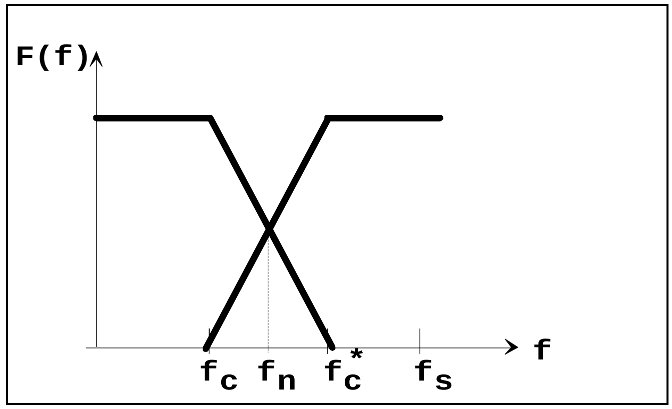


Figure 4. Frequency Relationships

Note that if there is energy at f_s or any integer multiple of f_s that it will alias to zero frequency. To ensure that the fundamental interval is not distorted, we must ensure that there is no detectable energy beyond $f = f_s - f_c$. A review of Figure 4 will verify that this frequency is f_c^*. Since we have chosen f_s based on f_c^*, we are sure that there is no distortion in the fundamental interval.

EXAMPLE 1

It is desired to measure a process variable which has a bandwidth of 10Hz (f_c = 10) using a system with a two-pole (i.e., 12dB/Octave) filter with cutoff frequency equal to 10Hz. Calculate the required sampling rate f_s such that the maximum possible distortion introduced by aliasing is 1%.

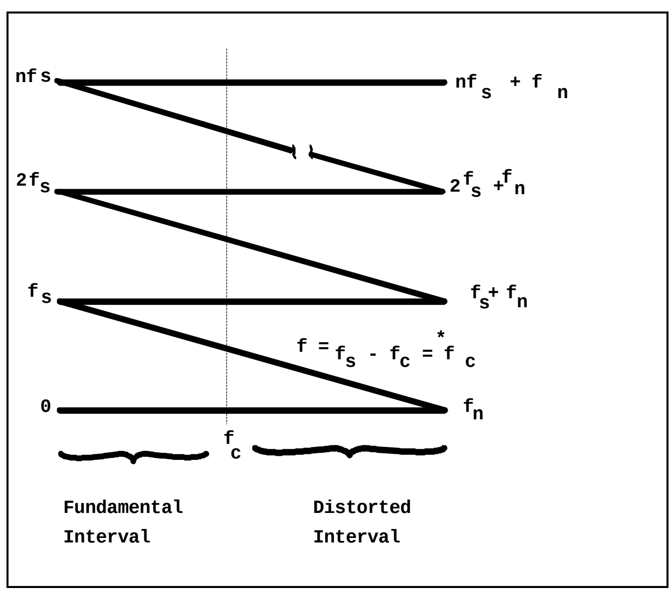


Figure 5. Frequency Folding Diagram

Calculate the frequency f_c^* beyond which all energy is diminished by the distortion specification.

- Number of octaves, N:

N = (Desired Attenuation, dB) / (Filter Rolloff Rate, dB/Octave)

where Desired Attenuation is 1%. Expressing this in dB:

$$dB = 20 \log (0.01) = -40$$

Thus

$$N = (-40dB) / (-12dB/Octave) = 3.3$$

- The frequency, f_c^*, corresponding to this attenuation is:

$$f_c^* = 2^N f_c = 100.8Hz$$

- The folding frequency, f_n, is:

$$f_n = ½ (f_c + f_c^*) = 55.4Hz$$

- The sampling frequency, f_s , is:

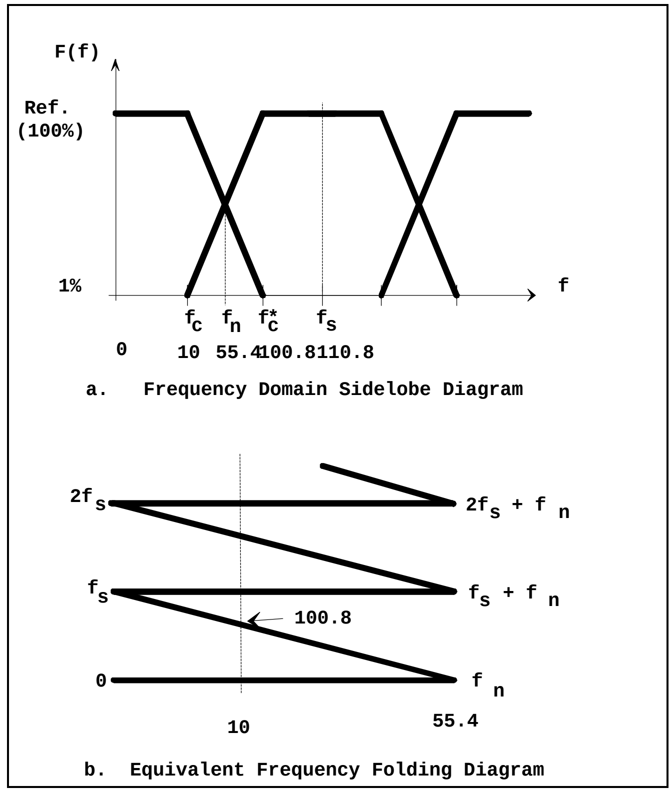
$$f_s \geq 2f_n \geq 110.8Hz$$

Graphical Interpretation

The sketches above illustrate that 10Hz can contain 1% distortion. However, for this to occur there must be energy at f = 100.8Hz which aliases to f = 10Hz.

EXAMPLE 2

For the above example, calculate the required sampling rate if a four-pole filter (24dB/Octave) is used rather than a two-pole filter.



a. Frequency Domain Sidelobe Diagram

b. Equivalent Frequency Folding Diagram

Solution

- Number of Octaves, N:

$$N = (-40dB) / (-24dB/Octave) = 1.6$$

- $f_c^* = 2^N f_c = 31.7Hz$

- Folding frequency, f_n :

$$f_n = ½ (f_c + f_c^*) = 20.9Hz$$

- Sampling frequency, f_s :

$$f_s \geq 2f_n \geq 41.8Hz$$

Similarly, if we had used a six-pole filter (36dB/Octave), the required sampling rate would be 31.6Hz.

SECTION 4.0
PROCESSING SAMPLED DATA

Although we have taken care in choosing f_s such that no distortion has been introduced in the fundamental interval, any energy that lies between f_c and f_n in the original spectrum will still be present in the sampled results. Because of the pre-sampler filter (often termed anti-alias filter), this energy will be diminished in amplitude according to its relative position to the filter cutoff frequency f_c and the filter's attenuation characteristics. Thus, the attenuation at a frequency f_x which lies between f_c and f_n can be computed as:

Attenuation at f_x, dB =
(Rolloff characteristics) [log (f_x /f_c)]/log(2)]

where the Rolloff Characteristics are the filter's attenuation expressed in terms of dB/Octave.

Since the sampled data will contain energy at unwanted frequencies, it is necessary that the discrete samples be further processed through some type of digital filter. If we simply average the sampled data over some time interval ❑T, then there are three consequences we must consider. These are:

1. The effective bandwidth will be reduced to BW = 1/ 2ΔT

2. Any energy present from f_c to f_n may introduce an error in the average.

3. Any energy present which has an integer number of periods equal to ΔT will be effectively eliminated.

To illustrate digital filtering, two simple concepts are presented below.

4.1 Running Averages with Equally Weighted Samples

If we process the sampled data through a running average filter where all samples are equally weighted, the net effect is a low-pass filter which passes zero frequency and

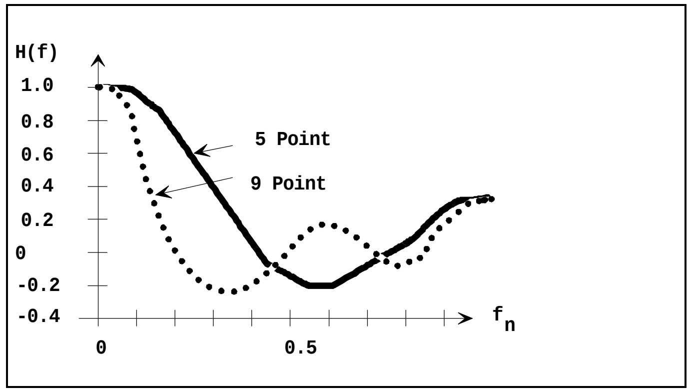


Figure 6. Transfer Function for Equally Weighted Running Averages

provides attenuation at all frequencies from zero to the folding frequency (f_n = f_s/2). The transfer function for this filter which uses n equally weighted samples is:

$$H(\alpha) = \mathrm{Sin}\,(n\omega/2)/n\mathrm{Sin}\,(\omega/2)$$

where $\omega = 2\pi\ (f_x/f_s)$. Figure 6 illustrates the transfer function for n = 5 and n = 9. It should be noted that the band of frequencies we are interested in is less than f_n. Ideally, we would like to design the digital filter such that the band of frequencies from zero to f_c is unattenuated and all frequencies greater than f_c are eliminated. Here again, this is the ideal case and a compromise must be made.

EXAMPLE 3

For Example 1 above using a two-pole filter with f = 10Hz, assume that the sampling frequency has been set at 110Hz (i.e., 110 samples per second per channel). As shown in the Example 1 sketches, any individual sample may contain energy caused by frequencies which lie anywhere from zero to f_n. However, since we are only interested in frequencies between zero and f_c, it is desirable to reduce any effects attributable to frequencies greater than f_c. For this case, process the data every 100mS through an 11-point equally weighted filter to produce a smoothed output each

100mS. Calculate the digital filter attenuation over the range of zero to 50 Hertz.

Solution

The transfer function for the 11-point average with equally weighted samples is:

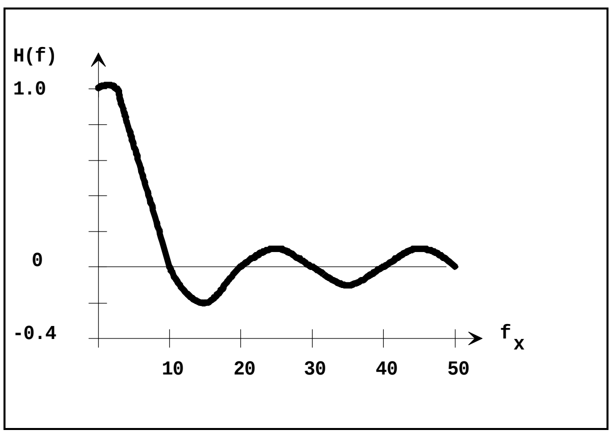
$$H(f) = \sin(n\omega/2)/n\sin(\omega/2)$$

where $\omega = 2\pi f_x / f_s$ with $f_s = 110$ and $n = 11$.

The response characteristics over the range are shown in the sketch below.

4.2 Running Averages with Unequal Weighted Samples

Using Least-Squares Quadratics as a filter design criterion, a low-pass filter similar to the above filter which uses equally weighted samples can be designed. As with the equally weighted samples filter, this filter also uses a running average. The samples, however, are



11-Point Equally Weighted Sample, f_s = 110 Hz

weighted unequally. The transfer function for the five-point running average is:

$$H(\omega) = (17 + 24\cos\omega - 6\cos 2\omega)/35$$

with $\omega = 2\pi(f_x/f_s)$. The coefficients (i.e., weights) are:

$$Y_n = \frac{1}{35}\left[-3x_{n-2} + 12x_{n-1} + 17x_n = 12x_{n+1} - 3x_{n+2}\right]$$

Figure 7 illustrates the transfer function. Note the higher degree of tangency at zero frequency as compared to Figure 6. There are other filter designs which extend this tangency even further.

The transfer function for the 11-point unequally weighted digital filter is:

$$H(\omega) = 89 + 168\cos(\omega) + 138\cos(2\omega) + 88\cos(3\omega) + 18\cos(4\omega) - 72\cos(5\omega)/429$$

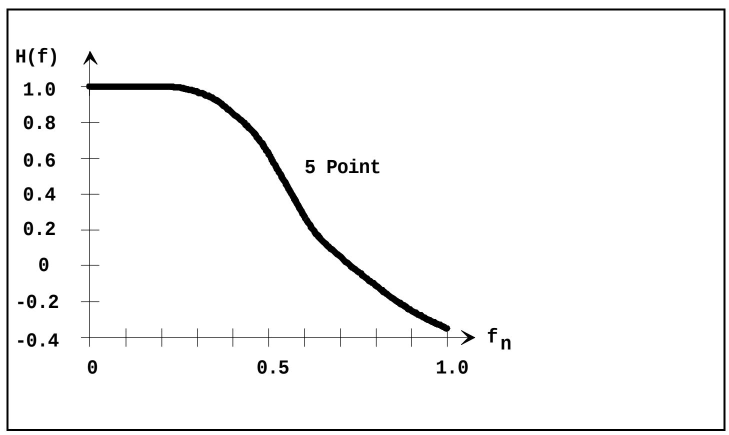


Figure 7. Transfer Function for Unequally Weighted Averages

SECTION 5.0
SUMMARY

Since information is lost when a continuum is represented by a finite number of samples, it is important that our samples be closely spaced. The space between samples (i.e., the value of the input function) can be approximated using any of various interpolating polynomials. By and large, this is the principal criterion we use in choosing sampling rate and various rules-of-thumb are offered based on this. Unfortunately, there is another aspect of selecting sampling rate which oftentimes is neglected.

Because of the non-uniqueness between equally spaced sampled data and the sampled function, the aliasing phenomenon may significantly distort our interpretation of the sampled results. As a consequence, it is critical that we select sampling rate based on the frequency beyond which there is no detectable energy. Since we are oftentimes unsure of this frequency, we must process the signal to be sampled through a low-pass hardware filter such that we can accurately control the bandwidth of the signal prior to sampling. Thus, rules-of-thumb regarding sampling are meaningless unless they specifically state the anti-alias filter's rolloff characteristics.

The techniques presented in Section 2 provides a mechanism for computing sampling rate based on desired bandwidth and filter characteristics. Using this, the tradeoff between different filters can be easily quantified. While systems that offer four and six-pole filters are initially more expensive than those with one or two-pole filters, they provide advantages which must be considered in assessing total costs. Improved filtering reduces sampling rate which relaxes aggregate throughput requirements, processor I/O bandwidth requirements, storage requirements and relaxes application software requirements.

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